Non-local Susceptibility for a Bounded Homogenized Wire Medium in the Spatial Domain

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Session: Homogenization
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Outline

• Introduction and Motivation
• Non-local Susceptibility for Finite Wire Medium
  ✓ Transport equation model
  ✓ Charge carrier reflection model
  ✓ The wave expansion method
• Spatial-Domain Formulations for Wire Medium Examples
• Conclusion
Spatially-Dispersive WM-Type Metamaterials

The wave expansion method

✓ Three different plane waves: TM, TE, TEM
✓ Additional boundary conditions (ABCs)
All –Angle Negative Refraction

Array of crossed wires

\[ \theta_t = \tan^{-1} \left( \frac{\Delta}{L} \right) \]

\( \theta_t \) (homog.) = \(-25^\circ\)

Full-wave simulations with the MoM

\( L = 10a \)

\( L = 21a \)

\( L = 31a \)

90 wires per plane

\( r_w = 0.05a \)

\( \omega a / c = 0.6 \)

\( \theta_t = -22^\circ \)

\( \theta_t = -25^\circ \)

\( \theta_t = -30^\circ \)

Negative Refraction with Mushroom Structure

\[ a = 2 \text{ mm}, \quad \varepsilon_h = 10.2, \quad g = 0.1a, \quad r = 0.025a, \quad \omega a/c = 0.45 \]


3D Finite Wire Media Topologies

Drift-diffusion model for wire medium

Isotropic Example: (Spherical Objects)

Anisotropic Example: (Cubical Object)

E. Forati and G. W. Hanson, IEEE TAP, 61, 3564-3574, July 2013

E. Forati and G. W. Hanson, Phys. Rev. B, 88, 125125, 2013
Local material

\[ P(r) = \varepsilon_0 \mathcal{K} \cdot E(r) = \varepsilon_0 (\varepsilon - 1) \cdot E(r) \]
\[ D(r) = \varepsilon_0 \varepsilon \cdot E(r) \]

Translationally invariant, homogeneous, non-local material

\[ P(r) = D(r) - \varepsilon_0 E(r) = \varepsilon_0 \int K(r - r') \cdot E(r') dr' \]

In the spatial transform domain \( r \leftrightarrow q \)

\[ P(q) = \varepsilon_0 \mathcal{K}(q) \cdot E(q) = \varepsilon_0 (\varepsilon(q) - 1) \cdot E(q) \]
\[ D(q) = \varepsilon_0 \varepsilon(q) \cdot E(q) \]
Material Boundary

For a finite material region

\[
D(r) = \varepsilon_0 \int \overline{\varepsilon}(r,r') \cdot E(r')dr'
\]

\[
P(r) = \varepsilon_0 \int \overline{\chi}(r,r') \cdot E(r')dr'
\]

The material response must take into account the material boundary – it cannot be derived from the bulk response!!!

**\( \overline{\chi}(r,r') \) as a Green’s function**

If \( P(r) \) and \( E(r) \) are related by a linear differential equation

\[
LP(r) = E(r)
\]

\[
B(P) = 0
\]

Then, \( \overline{Lg}(r,r') = 1\delta(r - r') \)

\[
B(\overline{g}) = 0
\]

Such that

\[
P(r) = \int \overline{g}(r,r') \cdot E(r')dr'
\]

We have

\[
\varepsilon_0 \overline{\chi}(r,r') = \overline{g}(r,r')
\]

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Homogenization of Wire Media

For long wavelengths the propagation of waves can be described using an effective medium approach

Tensor effective permittivity

In the FT domain \( z \leftrightarrow q_z \)

\[
\bar{\varepsilon}_{\text{bulk}} (q_z) = \varepsilon_h \varepsilon_0 \left( 1 + \hat{z}\hat{z} \frac{k_p^2}{q_z^2 - k_h^2} \right)
\]

\[
k_h = k_0 \sqrt{\varepsilon_h} = \omega \sqrt{\mu_0 \varepsilon_0 \varepsilon_h}
\]

\[
(\frac{k_p p}{\omega})^2 \approx \frac{2\pi}{\ln \left( \frac{p^2}{4r(a-r)} \right)} \approx 2\pi / \ln \left( \frac{p}{2\pi r} \right) + 0.5275
\]


Homogenization of Wire Media

Local, bound-charge polarization response of the background

\[ \mathbf{P}^{\text{pol}}(\rho, q_z) = \varepsilon_0 \mathbf{\overline{\chi}}^{\text{pol}}_{\text{bulk}} \cdot \mathbf{E}(\rho, q_z) = \varepsilon_0 (\varepsilon_h - 1) \cdot \mathbf{E}(\rho, q_z) \]

Non-local conductive polarization response

\[ \mathbf{P}^{\text{cond}}(\rho, q_z) = \mathbf{D}(\rho, q_z) - \varepsilon_0 \varepsilon_h \mathbf{E}(\rho, q_z) = \varepsilon_0 \mathbf{\overline{\chi}}^{\text{cond}}_{\text{bulk}}(q_z) \cdot \mathbf{E}(\rho, q_z) \]

where

\[ \mathbf{\overline{\chi}}^{\text{cond}}_{\text{bulk}}(q_z) = \hat{\mathbf{\chi}} \varepsilon_h k_p^2 \frac{q_z - k_h^2}{q_z^2} \]

Using \( e^{-j\gamma|z|} / 2j\gamma \leftrightarrow (q_z^2 - \gamma^2)^{-1} \)

Spatial domain bulk conductive susceptibility

\[ \mathbf{\overline{\chi}}^{\text{cond}}_{\text{bulk}}(z - z') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{\overline{\chi}}^{\text{cond}}_{\text{bulk}}(q_z) e^{jq_z(z - z')} dq_z = \hat{\mathbf{\chi}} \varepsilon_h k_p^2 \frac{e^{-jk_h|z - z'|}}{2jk_h} \]
Model I: Transport Equation Model

Drift-diffusion model for a bulk medium

\[ \mathbf{J}^\text{cond}(\mathbf{r}) = \bar{\sigma}_0 \cdot \mathbf{E}(|\mathbf{r}|) + \left( \frac{1}{j\omega} \right) \mathbf{D}_0 \cdot \nabla \nabla \cdot \mathbf{J}^\text{cond}(\mathbf{r}) \]

\[ \bar{\sigma}_0 = \hat{\mathbf{z}} \cdot \mathbf{\sigma}_z \quad \text{‘local’ conductivity tensor} \]

\[ \mathbf{D}_0 = \hat{\mathbf{z}} \cdot \mathbf{D}_z \quad \text{diffusion coefficient tensor} \]

\[ \left( 1 - \frac{D_z}{j\omega} \frac{\partial^2}{\partial z^2} \right) J_z^\text{cond}(\mathbf{r}) = \sigma_z E_z(\mathbf{r}) \]

With \[ \sigma_z = D_z \varepsilon_h \varepsilon_0 k_p^2, \quad D_z = -\frac{j\omega}{k_h^2}, \quad J_z = j\omega p_z \]
we obtain the differential equation for polarization

\[ \left( 1 + \frac{1}{k_h^2} \frac{\partial^2}{\partial z^2} \right) P_z^\text{cond}(\mathbf{r}) = -\frac{k_p^2}{k_h^2} \varepsilon_h \varepsilon_0 E_z(\mathbf{r}) \]

We assume a generalized boundary condition

\[ \left( P_z^\text{cond}(z) + \alpha \frac{dP_z^\text{cond}(z)}{dz} \right) \bigg|_{z=\delta} = 0 \]

E. Forati and G. W. Hanson, IEEE TAP, 61, 3564-3574, July 2013

G. W. Hanson, E. Forati, M. G. Silveirinha, IEEE TAP, 60, 4219-4232, 2012
Green’s Function Problem

\[
\left( \frac{\partial^2}{\partial z^2} + k_h^2 \right) g(z, z') = -\delta(z - z')
\]
\[
\left( g(z, z') + \alpha \frac{dg(z, z')}{dz} \right) \bigg|_{z=\delta} = 0
\]

\[
P^\text{cond}_z(z) = k_p^2 \varepsilon_h \varepsilon_0 \int_{\delta}^{\infty} g(z, z') E_z(z') dz'
\]

We see that
\[
\chi^\text{cond}(z, z') = \varepsilon_h k_p^2 g(z, z')
\]

The usual Green’s function for local materials forms the boundary-dependent non-local susceptibility and the permittivity \( \varepsilon(z, z') = \varepsilon_h k_p^2 g(z, z') + \varepsilon_h \delta(z - z') \)

Susceptibility as the superposition of primary and scattered parts
\[
\chi^\text{cond}(z, z') = \chi^\text{cond}_p(z - z') + \chi^\text{cond}_s(z, z')
\]
\[
\left( \frac{\partial^2}{\partial z^2} + k_h^2 \right) \chi^\text{cond}_p(z - z') = -\frac{1}{\varepsilon_h k_p^2} \delta(z - z')
\]
\[
\left( \frac{\partial^2}{\partial z^2} + k_h^2 \right) \chi^\text{cond}_s(z, z') = 0
\]
The Green’s function is usually used to connect the electric field and current

\[
E(r) = \frac{1}{\varepsilon_0 \varepsilon_h} \left[ k_h^2 \mathbf{1} + \nabla \nabla \right] \cdot \int \mathbf{g}(r, r') \cdot \mathbf{P}(r') dr' + E^i(r)
\]

The relationship between the electric field and current/polarization is non-local via the superposition integral involving the Green’s function, albeit in a rather trivial (purely geometric) sense. However, a similar Green’s function provides the non-local combined material-geometric response \( \chi^{\text{cond}}(z, z') \) that accounts for both the material boundary and the material spatial dispersion!!!
Green’s Function for WM with Different ABCs

Finite WM slab of thickness $L$

$$g(0, z') = g(L, z') = 0$$

$$g(z, z') = \frac{1}{2jk_h} \left( e^{-jk_h|z-z'|} - e^{-jk_hz'} \cos(k_hz) \right. + \left. \left[ e^{-jk_hz'} \cot(k_hL) - e^{-jk_h(z-z')} \csc(k_hL) \right] \sin(k_hz) \right)$$

In the limit $L \to \infty$ and allowing small loss

$$g(z, z') = \frac{1}{2jk_h} \left( e^{-jk_h|z-z'|} - e^{-jk_h(z+z')} \right)$$

$$\chi^{\text{cond}}(z, z') = \frac{\varepsilon_h k_p^2}{2jk_h} \left( e^{-jk_h|z-z'|} - e^{-jk_h(z+z')} \right)$$

For the generalized ABC:

$$\left. \left( g(z, z') + \alpha \frac{dg(z, z')}{dz} \right) \right|_{z=0} = 0, \quad \left. \left( g(z, z') - \alpha \frac{dg(z, z')}{dz} \right) \right|_{z=L} = 0$$

As $L \to \infty$ we have

$$\chi^{\text{cond}}(z, z') = \varepsilon_h k_p^2 \left( \frac{e^{-jk_h|z-z'|}}{2jk_h} - \frac{1}{2jk_h} \left( 1 + \alpha jk_h \right) \frac{e^{-jk_h(z+z')}}{1 - \alpha jk_h} \right)$$

Model II: Charge Carrier Reflection Model

In the dead layer \( \overline{\chi}^{\text{cond}}(z, z') = \overline{\epsilon}_0, \quad 0 < z, z' < \delta \)

In the material half-space outside of the dead layer

\[
\overline{\chi}^{\text{cond}}(z, z') = \overline{\chi}_\text{bulk}(z - z') + \overline{\chi}_\text{bulk}(z + z' - 2\delta) \cdot \overline{\Upsilon}, \quad z, z' > \delta
\]

The non-local conductive polarization response

\[
P^{\text{cond}}(\mathbf{r}) = \overline{\epsilon}_0 \int_{\delta}^{\infty} \overline{\chi}^{\text{cond}}(z, z') \cdot \mathbf{E}(\rho, z') dz'
\]

If we ignore the dead layer, this model with \( \overline{\Upsilon} = \overline{0} \) is known as the ‘dielectric approximation’ (DA), wherein the bulk response is assumed to apply right up to the interface and where the interface does not affect the material response

\[
\overline{\chi}^{\text{cond}}(z, z') = \overline{\chi}_\text{bulk}(z - z')
\]


Boundary Condition

Charge carrier model leads directly to the ABC at the point \( z = \delta \)

For \( z = \delta \) and \( z' > \delta \) we have

\[
jk_h P_{z,\text{cond}} = \kappa \int_\delta^\infty (1 + U_{zz}) e^{-jk_h(z' - \delta)} E_z(z')dz'
\]

\[
\frac{\partial P_{z,\text{cond}}}{\partial z} = \kappa \int_\delta^\infty (1 - U_{zz}) e^{-jk_h(z' - \delta)} E_z(z')dz'
\]

\[
\kappa = \varepsilon_0 \varepsilon_h k_p^2 / 2
\]

By adding and subtracting the above two expressions we obtain the generalized ABC

\[
\left( P_{z,\text{cond}}(z) + \alpha \frac{dP_{z,\text{cond}}(z)}{dz} \right) \bigg|_{z=\delta} = 0
\]

where \( \alpha = -\frac{1}{jk_h (1-U_{zz})} \)

**Special cases**

\[
U_{zz} = 0 \text{ (DA)}
\]

\[
P_{z,\text{cond}} - \frac{1}{jk_h} \frac{\partial}{\partial z} P_{z,\text{cond}} = 0
\]

matched termination

\[
U_{zz} = -1
\]

\[
P_{z,\text{cond}} = 0
\]

termination with an insulator

\[
U_{zz} = +1
\]

\[
\frac{\partial}{\partial z} P_{z,\text{cond}} = 0
\]

termination with PEC

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M. G. Silveirinha, IEEE TAP, 54, 1766-1780, 2006
\[
\overline{\chi}^{\text{cond}}(z,z') = \frac{\varepsilon_h k_p^2}{2 j k_h} \hat{\mathbf{z}} \hat{\mathbf{z}} \left( e^{-j k_h |z-z'|} + U_{zz} e^{-j k_h (z+z' - 2 \delta)} \right), \quad z, z' > \delta
\]

The charge carrier reflection method result with \( U_{zz} = -1 \) is the same as the transport equation result, thus, if \( U_{zz} = -1 \) then solving the drift-diffusion equation and enforcing the ABC \( P_z^{\text{cond}} = 0 \) is equivalent to the charge carrier reflection model with \( U_{zz} = -1 \)

If \( \alpha \) is known we have \( U_{zz} = \frac{j k_h \alpha + 1}{j k_h \alpha - 1} \)

For a wire medium terminated with a thin metal surface (graphene) characterized by sheet conductivity \( \sigma_{2d} \)

\[
\alpha = \frac{\sigma_{2d}}{j \omega \varepsilon_0 \varepsilon_h} \quad U_{zz} = \frac{\sigma_{2d} \eta_h - 1}{\sigma_{2d} \eta_h + 1}
\]

\[
\eta_h = \sqrt{\frac{\mu_0}{\varepsilon_0 \varepsilon_h}}
\]

A. B. Yakovlev et al., IEEE TMTT, 59, 527-532, 2011
ABC and Green’s Function for Two Wire Mediums

\[
P_{1}^{\text{cond}}(\rho, z) = \varepsilon_0 \int_{-\infty}^{0} \overline{\chi}_{11}^{\text{cond}}(z, z') \cdot E_1(\rho, z')dz' + \varepsilon_0 \int_{0}^{\infty} \overline{\chi}_{12}^{\text{cond}}(z, z') \cdot E_2(\rho, z')dz'
\]

\[
P_{2}^{\text{cond}}(\rho, z) = \varepsilon_0 \int_{-\infty}^{0} \overline{\chi}_{21}^{\text{cond}}(z, z') \cdot E_1(\rho, z')dz' + \varepsilon_0 \int_{0}^{\infty} \overline{\chi}_{22}^{\text{cond}}(z, z') \cdot E_2(\rho, z')dz'
\]

Using the charge carrier reflection model (Model II)

\[
\overline{\chi}_{11}^{\text{cond}}(z, z') = \hat{\chi} \frac{\varepsilon_h k_p^2}{2jk_h} \left( e^{-jk_h|z-z'|} + U_{11} e^{jk_h(z+z'+2\delta)} \right)
\]

\[
\overline{\chi}_{12}^{\text{cond}}(z, z') = \hat{\chi} \frac{\varepsilon_h k_p^2}{2jk_h} U_{12} e^{jk_h(z-z'+2\delta)}
\]

\[
\overline{\chi}_{21}^{\text{cond}}(z, z') = \hat{\chi} \frac{\varepsilon_h k_p^2}{2jk_h} U_{21} e^{-jk_h(z-z'-2\delta)}
\]

\[
\overline{\chi}_{22}^{\text{cond}}(z, z') = \hat{\chi} \frac{\varepsilon_h k_p^2}{2jk_h} \left( e^{-jk_h|z-z'|} + U_{22} e^{-jk_h(z+z'-2\delta)} \right)
\]
Two ABCs are found

\[ P_{z1}^{\text{cond}} \left|_{\delta} - P_{z2}^{\text{cond}} \left|_{\delta} + \alpha \left( \frac{dP_{z1}^{\text{cond}}}{dz} \left|_{\delta} + \frac{dP_{z2}^{\text{cond}}}{dz} \left|_{\delta} \right) = 0 \right. \right. \]

\[ P_{z1}^{\text{cond}} \left|_{-\delta} + P_{z2}^{\text{cond}} \left|_{-\delta} + \beta \left( \frac{dP_{z1}^{\text{cond}}}{dz} \left|_{-\delta} - \frac{dP_{z2}^{\text{cond}}}{dz} \left|_{-\delta} \right) = 0 \right. \right. \]

where \( \alpha = \frac{1 + U_R - U_T}{jk_h (1 - U_R + U_T)} \), \( \beta = \frac{1 + U_R + U_T}{jk_h (1 - U_R - U_T)} \)

\[ U_{21} = U_{12} = U_T, \quad U_{11} = U_{22} = U_R \]

PEC: \( U_R = 1, \quad U_T = 0 \)

PMC: \( U_R = -1, \quad U_T = 0 \)

Thin conducting interface characterized with surface conductivity \( \sigma_{2d} \)

\[ U_R = \frac{\sigma_{2d} \eta_0}{\sigma_{2d} \eta_0 + 2\sqrt{\varepsilon_h}}, \quad U_T = \frac{2\sqrt{\varepsilon_h}}{\sigma_{2d} \eta_0 + 2\sqrt{\varepsilon_h}} \]
For transport equation model (Model I) we can form Green’s functions

Source is in region 1

\[ \left( \frac{\partial^2}{\partial z^2} + k_h^2 \right) g_{11}(z, z') = -\delta(z - z') \]

Source is in region 2

\[ \left( \frac{\partial^2}{\partial z^2} + k_h^2 \right) g_{12}(z, z') = -\delta(z - z') \]

\[ \left( \frac{\partial^2}{\partial z^2} + k_h^2 \right) g_{21}(z, z') = 0 \]

\[ \left( \frac{\partial^2}{\partial z^2} + k_h^2 \right) g_{12}(z, z') = 0 \]

Enforcing the ABCs on Green’s functions

\[ g_{11} \big|_{-\delta} - g_{21} \big|_{\delta} + \alpha \left( \frac{\partial g_{11}}{\partial z} \big|_{-\delta} + \frac{\partial g_{21}}{\partial z} \big|_{\delta} \right) = 0 \]

\[ g_{12} \big|_{-\delta} - g_{22} \big|_{\delta} + \alpha \left( \frac{\partial g_{12}}{\partial z} \big|_{-\delta} + \frac{\partial g_{22}}{\partial z} \big|_{\delta} \right) = 0 \]

\[ g_{11} \big|_{-\delta} + g_{21} \big|_{\delta} + \beta \left( \frac{\partial g_{11}}{\partial z} \big|_{-\delta} - \frac{\partial g_{21}}{\partial z} \big|_{\delta} \right) = 0 \]

\[ g_{12} \big|_{-\delta} + g_{22} \big|_{\delta} + \beta \left( \frac{\partial g_{12}}{\partial z} \big|_{-\delta} - \frac{\partial g_{22}}{\partial z} \big|_{\delta} \right) = 0 \]

This leads to the same susceptibilities obtained with the charge carrier reflection method

\[ \chi_{ij}^{\text{cond}}(z, z') = \epsilon_h k_p^2 g_{ij}(z, z') \]
Model III: The Wave Expansion Method

Wire medium supports three types of modes

- **TE-z (ordinary mode)**, the ordinary mode does not interact with the wires
  \[
  k_{z,TE} = \sqrt{(\omega/c)^2 \varepsilon_h^2 - k_x^2 - k_y^2}
  \]

- **TM-z (extraordinary mode)**, the extraordinary mode interacts with the wires, and corresponds to nonzero currents in the wires and nonzero electric field along the wires
  \[
  k_{z,TM} = -j\sqrt{k_p^2 + k_x^2 + k_y^2 - (\omega/c)^2 \varepsilon_h^2}
  \]

- **TEM (transmission line mode)**, corresponds to nonzero currents in the wires and zero electric field along the wires, and is defined for any wave vector in the transverse direction
  \[
  k_{z,TEM} = (\omega/c)\sqrt{\varepsilon_h}
  \]

In the spatial domain, the fields in wire medium satisfy the following system of equations subject to usual boundary conditions and ABCs at WM interface

\[
\nabla \times \mathbf{E} = -j\omega\mu_0 \mathbf{H}
\]

\[
\nabla \times \mathbf{H} = j\omega\varepsilon_0 \mathbf{E} + j\omega P_z \hat{z}
\]

\[
\left(k_0^2 + \frac{d^2}{dz^2}\right)P_z = -\varepsilon_0 k_p^2 E_z
\]
**Reflection from a Uniaxial Half-Space WM**

**Model II** (charge carrier reflection model)

\[
P_{z}^{\text{cond}}(z) = \int_{0}^{\infty} dz' \frac{k_p^2}{2 jk_0} \left( e^{-jk_0|z-z'|} + U_{zz} e^{-jk_0(z+z')} \right) e^{-jk_h(z'-\delta)} \times \left[ k_0^2 + \frac{\partial^2}{\partial z'^2} \right] \int_{0}^{\infty} \frac{e^{-p|z'-z''|}}{2p} P_{z}^{\text{cond}}(z'') \, dz'' + \varepsilon_0 E_z^i(z') \right)
\]

\[
U_{zz} = -1, \quad p = \sqrt{q_\rho^2 - k_0^2}, \quad q_\rho^2 = q_x^2 + q_y^2
\]

It involves a two-fold semi-infinite integral and it is time consuming to compute.

**Model I** (transport equation model)

It solves directly the differential equation for polarization as an integral-differential equation in the \( q_\rho \) transform domain

\[
- \frac{k_0^2}{k_p^2} \left( 1 + \frac{1}{k_0^2} \frac{\partial^2}{\partial z^2} \right) P_{z}^{\text{cond}}(z) = \left[ k_0^2 + \frac{\partial^2}{\partial z^2} \right] \int_{0}^{\infty} \frac{e^{-p|z-z'|}}{2p} P_{z}^{\text{cond}}(z') \, dz' + \varepsilon_0 E_z^i(z)
\]

It involves a single semi-infinite integral to compute.
Reflection from a Uniaxial Half-Space WM

Model III (the wave expansion method)

It involves solving for bulk waves that exist in each region and enforcing the usual boundary conditions along with the ABC

With TM wave incidence, the fields for $z < 0$

$$H_y = (e^{-\gamma_0 z} - Re^{\gamma_0 z}) e^{-j\gamma z}$$

$$E_x = -\frac{1}{j\omega\varepsilon_0} \frac{\partial H_y}{\partial z} = \frac{\gamma_0}{j\omega\varepsilon_0} (e^{-\gamma_0 z} + Re^{\gamma_0 z}) e^{-j\gamma z}$$

$$E_z = \frac{1}{j\omega\varepsilon_0} \frac{\partial H_y}{\partial x} = -\frac{j\gamma}{j\omega\varepsilon_0} (e^{-\gamma_0 z} - Re^{\gamma_0 z}) e^{-j\gamma z}$$

Here, $\gamma_0 = \sqrt{q_x^2 - k_0^2}$, $\gamma_{TM} = \sqrt{q_x^2 + k_p^2 - k_0^2}$, $\gamma_{TEM} = jk_0$

Enforcing the ABC $P_z^\text{cond} = 0$ at $z = 0$ and matching tangential fields results in

$$q_x^2 B_{TEM} = k_p^2 A_{TM}, \quad R = \frac{(\gamma_{TM} - \gamma_0)(\gamma_{TEM} - \gamma_0)}{(\gamma_{TM} + \gamma_0)(\gamma_{TEM} + \gamma_0)}$$

$$A_{TM} = \frac{2q_x^2 \gamma_0}{(k_p^2 + q_x^2)\gamma_0 + q_x^2 \gamma_{TM} + k_p^2 \gamma_{TEM}}$$

It can be shown that these fields satisfy

$$\nabla \times E = -j\omega\mu_0 H$$

$$\nabla \times H = j\omega\varepsilon_0 E + \hat{z} \frac{k_p^2}{2\eta_0} \int_0^\infty (e^{-jk_0|z-z'|} - e^{-jk_0(z+z')})E_z(z')dz'$$

The fields for $z > 0$

$$H_y = (A_{TM} e^{-\gamma_{TM} z} + B_{TEM} e^{-\gamma_{TEM} z}) e^{-j\gamma z}$$

$$E_x = \frac{1}{j\omega\varepsilon_0} (\gamma_{TM} A_{TM} e^{-\gamma_{TM} z} + \gamma_{TEM} B_{TEM} e^{-\gamma_{TEM} z}) e^{-j\gamma z}$$

$$E_z = -\frac{\eta_0 (k_p^2 + q_x^2)}{k_0 q_x} A_{TM} e^{-\gamma_{TM} z} e^{-j\gamma z}$$

And this solution is equivalent to the Model I and Model II.
Isotropic Wire Medium

Comparison of transport equation and wave expansions models

Isotropic connected wire medium

Wire medium slab

E. Forati and G. W. Hanson, IEEE TAP, 61, 3564-3574, July 2013
Isotropic Wire Medium

Comparison of transport equation and wave expansions models

Transmission coefficient


E. Forati and G. W. Hanson, IEEE TAP, 61, 3564-3574, July 2013

slab thickness: 276 nm
wire radius: 8.25 nm
wire period: 276 nm
frequency: 76.1 THz
wire permittivity: -810-j50
host permittivity: 1
Isotropic Wire Medium

slab thickness: 276 nm
wire radius: 8.25 nm
wire period: 276 nm
frequency: 76.1 THz
wire permittivity: -810-j50
host permittivity: 11.9

Internal fields of a wire medium slab

E. Forati and G. W. Hanson, IEEE TAP, 61, 3564-3574, July 2013
Conclusions

✓ We have shown that when the polarization and electric field are related by a linear differential equation the non-local susceptibility \( \chi(r, r') \) for a non-translationally invariant wire medium is given by a Green’s function related to the material geometry.

✓ We have also shown that two previous methods (transport equation model and wave expansion model) for solving wave interaction problems for bounded wire media are equivalent to each other, and to a third method involving particle reflection at the boundary.