SCATTERED FIELD FDTD FORMULATIONS FOR DEBYE DISPERSIVE MEDIA


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Abstract --- New implementation of scattered field FDTD formulations of dispersive material are employed yielding accurate and mathematically compact procedures. This work is based on the application of the Z-transform on the Debye model representing the dispersion in conjunction with the relation between the electric field and the displacement current. The result is an approximate frequency domain formulation, which can be easily transformed into discrete time domain updating equations. Three approaches are presented. The last one is mathematically compact, easy to use, and can be extended to any number of Debye poles without the need for further mathematical manipulations. Reflection coefficient at the interface between the air and the dispersive medium having more than one pole is computed and compared with the analytical solution.

I. INTRODUCTION

The finite difference time domain is a versatile numerical technique that can be used in the analysis and design of different electromagnetic problems in the time domain. Introducing the dispersive properties in the FDTD technique is not a direct manner compared to the frequency domain numerical techniques. There are two different categories that can describe the problem, the recursive convolution approach [1], and the auxiliary differential equation (ADE) based methods [3-7]. The first category of analysis aims to less memory usage. The second category moves toward giving a more accurate result but on the expense of the required memory. The Z-transform technique can be used in conjunction with the ADE methods to produce mathematically compact FDTD updating equations. In this paper three approaches are introduced based on the Z-transform.

II. THE FIRST APPROACH

This approach is an extension of the formally developed technique by Beggs [5] to be employed in FDTD scattering codes, where only one auxiliary equation is used. It relates both the displacement current and the electric field through higher order differentiation, which is approximated using the Z-transform approximation techniques. These approximations are backward difference and bilinear schemes. The electric field is divided into incident and scattered components. The displacement current is then calculated from the magnetic field using Ampere’s law while considering the relation between the incident electric and magnetic fields. The following Debye model describes the permittivity in frequency domain as,

$$\epsilon(\omega) = \epsilon_{\infty} + \sum_{p=1}^{P} \frac{\Delta\epsilon_p}{1 + j\omega\tau_p}, \quad p=1,2,\ldots,P$$

(1)

For two poles, \(p=2\), equation (1) can be written as

$$\epsilon(\omega) = \epsilon_e + j\omega(\epsilon_{s1}\tau_2 + \epsilon_{s2}\tau_1) - \omega^2\tau_1\tau_2$$

$$\epsilon_\tau = \epsilon_{s1} + \epsilon_{s2} - \epsilon_{\infty}.$$  

(2)

(3)

The displacement current \(D(\omega)\) is related to the electric field \(E(\omega)\) through the following equation

$$D(\omega) = \epsilon_0 \mu(\omega) E(\omega)$$

(4)

which can be transformed to the discrete time domain through backward difference or bilinear transformation schemes. Substituting for \(E(\omega)\) from equation (2) into equation (4) yields the displacement current

$$D(\omega) \left[ 1 + j\omega(\tau_1 + \tau_2) + (j\omega)^2 \tau_1\tau_2 \right]$$

$$= \epsilon_0 \left[ \epsilon_e + j\omega(\epsilon_{s1}\tau_2 + \epsilon_{s2}\tau_1) + \tau_1\tau_2 (j\omega)^2 \right] E(\omega)$$

(5)

In the following subsections the two proposed known approximations in the Z-domain are presented along with the development of the updating time domain equations.

a) BACKWARD DIFFERENCE APPROXIMATION

The transformation from the frequency to time domain through Z-domain follows the following relations
\[ j\omega f \Rightarrow \frac{1 - z^{-1}}{\Delta t} f \Rightarrow \frac{f^{n+1} - f^n}{\Delta t} \] (6)

\[ (j\omega)^2 f \Rightarrow \frac{1 - 2z^{-1} + z^{-2}}{\Delta t^2} \Rightarrow \frac{f^{n+1} - 2f^n + f^{n-1}}{\Delta t^2}. \] (7)

Using equations (6) and (7), equation (5) becomes

\[ D^{n+1} + (\tau_1 + \tau_2) \frac{D^{n+1} - D^n}{\Delta t} + \tau_1 \tau_2 \frac{D^{n+1} - 2D^n + D^{n-1}}{\Delta t^2} = \epsilon_0 \epsilon_s E^{n+1} + \epsilon_s (\epsilon_{s1} \tau_2 + \epsilon_{s2} \tau_1) \frac{E^{n+1} - E^n}{\Delta t} \]

\[ + \epsilon_0 \tau_1 \tau_2 \frac{E^{n+1} - 2E^n + E^{n-1}}{\Delta t^2}. \] (8)

Thus the electric field updating equation reduces to

\[ E^{n+1} = \left( \frac{cd2}{cel} \right) E^n - \epsilon_0 \left( \frac{ced}{cel} \right) E^n + \left( \frac{cd1}{cel} \right) D^{n+1} \]

\[ - \left( \frac{cd2}{cel} \right) D^n + \left( \frac{ced}{cel} \right) D^{n-1} \] (9)

where

\[ ced = \tau_1 \tau_2 / \Delta t^2, \]

\[ cel = \epsilon_0 \left[ \epsilon_s + \frac{(\epsilon_{s1} \tau_2 + \epsilon_{s2} \tau_1)}{\Delta t} + ced \right], \]

\[ ce2 = \epsilon_0 \left[ \frac{(\epsilon_{s1} \tau_2 + \epsilon_{s2} \tau_1)}{\Delta t} + 2ced \right], \]

\[ cd1 = 1 + \frac{\tau_1 + \tau_2}{\Delta t} + ced, \]

\[ cd2 = \frac{\tau_1 + \tau_2}{\Delta t} + 2ced. \]

b) BILINEAR TRANSFORMATION

The transformation from the frequency domain to the Z-domain is done through

\[ j\omega \Rightarrow \frac{2}{\Delta t} \left[ 1 - z^{-1} \right]. \] (10)

Applying (10) on both sides of equation (5) gives,

\[ D(z) + \frac{2(\tau_1 + \tau_2)}{\Delta t} \left[ 1 - z^{-1} \right] D(z) + \frac{4\tau_1 \tau_2}{\Delta t} \left[ 1 - 2z^{-1} + z^{-2} \right] D(z) = \epsilon_0 \epsilon_s E(z) + \frac{2\epsilon_0 (\epsilon_{s1} \tau_2 + \epsilon_{s2} \tau_1)}{\Delta t} \left[ 1 - z^{-1} \right] E(z) \]

\[ + \frac{4\epsilon_0 \tau_1 \epsilon_s \epsilon_w}{\Delta t^2} \left[ \frac{1 - 2z^{-1} + z^{-2}}{1 + 2z^{-1} + z^{-2}} \right] E(z). \] (11)

Fig. 1 Reflection coefficient computed using the BD-scheme at the interface between air and a second order Debye model.

Upon multiplying both sides of equation (11) by \( 1 + 2z^{-1} + z^{-2} \) and considering that the multiplication by \( z^{-1} \) in the Z-domain yields a backward shift one time step, the electric field updating equation becomes,

\[ E^{n+1} = \left( \frac{cd1}{cel} \right) D^{n+1} + \left( \frac{cd2}{cel} \right) D^n + \left( \frac{cd3}{cel} \right) D^{n-1} - \left( \frac{ce2}{cel} \right) \left[ \frac{ce3}{cel} \right] E^n + \left( \frac{ce3}{cel} \right) E^{n-1} \] (12)

where

\[ cel = \epsilon_0 \epsilon_s + \frac{2\epsilon_0 (\epsilon_{s1} \tau_2 + \epsilon_{s2} \tau_1)}{\Delta t} + \frac{4\epsilon_0 \tau_1 \tau_2 \epsilon_w}{\Delta t^2}, \]

\[ ce2 = 2\epsilon_0 \epsilon_s - \frac{8\epsilon_0 \tau_1 \tau_2 \epsilon_w}{\Delta t^2}, \]

\[ ce3 = \epsilon_0 \epsilon_s - \frac{2\epsilon_0 (\epsilon_{s1} \tau_2 + \epsilon_{s2} \tau_1)}{\Delta t} + \frac{4\epsilon_0 \tau_1 \tau_2 \epsilon_w}{\Delta t^2}, \]

\[ cd1 = 1 + \frac{2(\tau_1 + \tau_2)}{\Delta t} + \frac{4\tau_1 \tau_2}{\Delta t} \left[ 1 - 2z^{-1} + z^{-2} \right], \]

\[ cd2 = \frac{8\tau_1 \tau_2}{\Delta t} - \frac{8\tau_1 \tau_2 \epsilon_w}{\Delta t^2}. \]

III. THE SECOND APPROACH

This method is based on the transformation of the Debye poles into Z-domain. Only one auxiliary equation, which
relates the displacement current vector to the electric field is used. It is developed for both scattering and total field formulations. Considering the case for two Debye poles, equation (1) can be written as

\[ \epsilon(z) = \epsilon_0 \frac{\Delta \epsilon_1}{1 + j \omega \tau_1} + \frac{\Delta \epsilon_2}{1 + j \omega \tau_2} \]

Applying the Z-transform to equation (13) gives

\[ \mathcal{Z}\{ \epsilon(z) \} = \epsilon_0 \left[ \frac{\Delta \epsilon_1 \Delta t / \tau_1}{1 - e^{-\Delta t / \tau_1}} + \frac{\Delta \epsilon_2 \Delta t / \tau_2}{1 - e^{-\Delta t / \tau_2}} \right] \]

\[ \mathcal{Z}\{ \epsilon(z) \} = \epsilon_0 \left[ \frac{\Delta \epsilon_1 \left(1 - e^{-\Delta t / \tau_1}\right) \Delta t / \tau_1 + \Delta \epsilon_2 \left(1 - e^{-\Delta t / \tau_2}\right) \Delta t / \tau_2}{1 - e^{-\Delta t / \tau_1}} + \frac{\Delta \epsilon_2 \left(1 - e^{-\Delta t / \tau_2}\right) \Delta t / \tau_2}{1 - e^{-\Delta t / \tau_2}} \right] \]

\[ D(z) = \epsilon(z) \mathcal{Z}\{ \epsilon(z) \} \]

which can be written as,

\[ D(z) \left[ 1 - \left( e^{-\Delta t / \tau_1} + e^{-\Delta t / \tau_2} \right) z^{-1} + e^{-\left(\Delta t / \tau_1 + \Delta t / \tau_2\right)} \right] = \epsilon_0 \left[ \epsilon_0 + \Delta \epsilon_1 \Delta t / \tau_1 + \Delta \epsilon_2 \Delta t / \tau_2 \right] \mathcal{Z}\{ \epsilon(z) \}

\[ = \epsilon_0 \epsilon_0 \left( e^{-\Delta t / \tau_1} + e^{-\Delta t / \tau_2} \right) z^{-1} \mathcal{Z}\{ \epsilon(z) \}

\[ - \left( \Delta \epsilon_1 \Delta t / \tau_1 \right) e^{-\Delta t / \tau_1} z^{-1} \mathcal{Z}\{ \epsilon(z) \}

\[ - \left( \Delta \epsilon_2 \Delta t / \tau_2 \right) e^{-\Delta t / \tau_2} z^{-1} \mathcal{Z}\{ \epsilon(z) \} \]

while the displacement current in the Z-domain is given by

\[ + \epsilon_0 \epsilon_0 e^{-\left(\Delta t / \tau_1 + \Delta t / \tau_2\right)} z^{-2} \mathcal{Z}\{ \epsilon(z) \}. \]

Transforming (17) into discrete time domain, yields

\[ E_{n+1} = \left( \frac{ce2}{ce1} \right) E^n - \left( \frac{ce3}{ce1} \right) E_{n-1} + \frac{D_{n+1}}{ce1} D^n + \frac{cd2}{ce1} D_{n-1} - E_i. \]

The above equation is for total field formulation. Upon representing the total field in terms of incident and scattered fields, the corresponding scattered electric field updating expression is given by

\[ E_{s f}^{n+1} = \left( \frac{ce2}{ce1} \right) (E_s^n + E_i^n) - \left( \frac{ce3}{ce1} \right) (E_{s f}^{n-1} - E_i^{n-1}) \]

\[ + \frac{D_{s f}^{n+1}}{ce1} D^n + \frac{cd2}{ce1} D_{s f}^{n-1} - E_i^{n+1} \]

where \( cd1 = e^{-\Delta t / \tau_1} + e^{-\Delta t / \tau_2} \), \( cd2 = e^{-\left(\Delta t / \tau_1 + \Delta t / \tau_2\right)} \), \( ce1 = \epsilon_0 \left[ \epsilon_0 + \Delta \epsilon_1 \Delta t / \tau_1 + \Delta \epsilon_2 \Delta t / \tau_2 \right] \), \( ce2 = -\epsilon_0 \epsilon_0 \left[ e^{-\Delta t / \tau_1} + e^{-\Delta t / \tau_2} \right] + \left( \Delta \epsilon_1 \Delta t / \tau_1 \right) e^{-\Delta t / \tau_1} \]

\[ - \epsilon_0 \left[ \left( \Delta \epsilon_2 \Delta t / \tau_2 \right) e^{-\Delta t / \tau_2} \right], \]

\[ ce3 = \epsilon_0 \epsilon_0 e^{-\left(\Delta t / \tau_1 + \Delta t / \tau_2\right)} \]

III. THE THIRD APPROACH

In this approach the auxiliary equation between the electric and displacement fields given by equation (16) is used, however, the permittivity of the dispersive media is represented by additional real variables that yield a set of additional auxiliary equations, such that
\[ D(z) = \varepsilon_0 \varepsilon_\infty E(z) + \sum_{p=1}^{P} I_p(z) \]  
(20)

where \[ I_p(z) = \frac{\varepsilon_0 \Delta \varepsilon_p \Delta t / \tau_p}{1 - e^{-(\Delta t / \tau_p) Z^{-1}}} E(z) \].  
(21)

\[ I_p \] is the additional real variables assigned to each Debye pole.

\[ 1 - e^{-(\Delta t / \tau_p) Z^{-1}} I_p(z) = \varepsilon_0 \Delta \varepsilon_p \left( \Delta t / \tau_p \right) E(z) \]  
(22)

Transforming equations (21) and (20) into discrete time domain leads to,

\[ I_p^{n+1} = e^{-(\Delta t / \tau_p)} I_p^n - \varepsilon_0 \Delta \varepsilon_p \left( \Delta t / \tau_p \right) E^{n+1} \]  
(23)

\[ D^{n+1} = \varepsilon_0 \varepsilon_\infty E^{n+1} + \sum_{p=1}^{P} I_p^{n+1} \].  
(24)

Substituting (23) into (24) leads to the updating equation for the electric field based on the total field formulation,

\[ E^{n+1} = (1/ce) D^{n+1} - \sum_{p=1}^{P} (ce_p / ce) I_p^{n} \].  
(25)

The corresponding expression for the scattered electric field is then given by,

\[ E_s^{n+1} = (1/ce) D^{n+1} - \sum_{p=1}^{P} (ce_p / ce) \]  
(26)

with \( ce = \varepsilon_0 \left[ \varepsilon_\infty + \sum_{p=1}^{P} \Delta \varepsilon_p \left( \Delta t / \tau_p \right) \right] \), \( ce_p = e^{-(\Delta t / \tau_p)} \).

VI. NUMERICAL RESULTS

In order to verify the accuracy of the developed methods, wide band one-dimensional reflection coefficients based on FDTD calculations are performed along with the calculations based on the exact analytical solution. The FDTD results compared with those based on the exact solution are shown in Figs. 1 to 4 for the three methods presented.

V. CONCLUSION

The formulation, numerical performance and comparison of these three approaches are presented, the first and the second approaches need involved mathematical treatment for higher order poles. Therefore, we presented formulations with only second order Debye poles as a demonstration of the techniques. Extension to higher order poles is a straightforward procedure. The third approach is highly recommended. It yields good agreement with the analytical solution and doesn’t need any additional treatment for higher order poles. Two and three order Debye poles are tested using the third approach. Extension to any higher order poles is straightforward and can be easily implemented. Only one-dimensional test cases are introduced, however the presented formulations are valid for two and three-dimensional problems.

REFERENCES


