A FDTD SCATTERED FIELD FORMULATION FOR DISPERSIVE MEDIA

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Abstract

This summary presents a scattered field formulation for the finite difference time domain technique applied to dispersive media characterized by Debye equation with two relaxation constants. The 3D formulation is based on the auxiliary differential equation (ADE) approach. For verification purposes, numerical results for a one dimensional problem involving reflections from a dispersive dielectric half-space are presented and compared with the exact solution.

Introduction

The finite difference time domain (FDTD) method has been applied successfully to a wide variety of problems including complex interaction of electromagnetic fields with frequency independent materials. Practical applications, however, always include frequency depended material that exhibit dispersion. The solution to those problems is usually based on one of the following two methods. The first method is to convert the complex permittivity of the material from the frequency domain to the time domain and convolve it with the time domain electric fields to obtain time domain fields for dispersive media [1]. This approach is usually called recursive convolution (RC) and has been efficiently applied to materials described by first order Debye relaxation equation and a second order Lorentz equation with multiple poles. The RC method is usually implemented with the total field formulation of the FDTD algorithm [2-3]. A scattered field formulation for the RC method is presented in 1993 with no further developments [4]. The second method is based on the use of an auxiliary differential equation relating the electric flux density to the electric field in the time domain along with the standard FDTD updating equations. Thus it is called the auxiliary differential equation (ADE) method [5-6]. This method requires a bit more memory than the RC method, but usually results in improved accuracy. The ADE has been applied with the total field formulation of the FDTD algorithm. However, because of the many advantages of using the scattered field over the total field formulation, this paper presents a scattered field formulation for a dispersive media based on the ADE method.
dispersive relation is based on Debye equation with two relaxation constants which is general enough to cover many practical applications.

Formulation

This method is a direct solution of the differential form of Faraday’s and Ampere’s laws
\[
\nabla \times \overrightarrow{E} = -\mu \frac{\partial \overrightarrow{H}}{\partial t}, \quad \nabla \times \overrightarrow{H} = \frac{\partial \overrightarrow{D}}{\partial t}
\]
(1)

where the displacement vector is related to the electric field through the complex permittivity by \( \overrightarrow{D} = \varepsilon^*(\omega) \overrightarrow{E} \). When writing Maxwell’s equation in the above form, both the conduction and displacement currents are combined in the term \( \frac{\partial \overrightarrow{D}}{\partial t} \) or in the definition of the complex permittivity. The complex permittivity of the medium can be defined by a two relaxation Debye expression, such that
\[
\varepsilon^*(\omega) = \varepsilon_0 \left[ \varepsilon_s + \frac{\varepsilon_{s1} - \varepsilon_\infty}{1 + j\omega\tau_1} + \frac{\varepsilon_{s2} - \varepsilon_\infty}{1 + j\omega\tau_2} \right]
\]
(2)

which can be simplified to
\[
\varepsilon^*(\omega) = \varepsilon_0 \left[ \varepsilon_s + j\omega (\varepsilon_{s1}\tau_2 + \varepsilon_{s2}\tau_1) - \omega^2 \tau_1\tau_2 \varepsilon_\infty \right] \frac{1}{1 + j\omega (\tau_1 + \tau_2) - \omega^2 \tau_1\tau_2}
\]

where \( \varepsilon_s = \varepsilon_{s1} + \varepsilon_{s2} - \varepsilon_\infty \).

Assuming that \( \exp(j\omega t) \) is the time dependence, the displacement-electric field vector relation, for such a dispersive media, can be written as a differential equation in time domain as follows
\[
\tau_1\tau_2 \frac{\partial^2 \overrightarrow{D}}{\partial t^2} + (\tau_1 + \tau_2) \frac{\partial \overrightarrow{D}}{\partial t} + \overrightarrow{D} = \varepsilon_0 \left[ \varepsilon_s \overrightarrow{E} + (\varepsilon_{s1}\tau_2 + \varepsilon_{s2}\tau_1) \frac{\partial \overrightarrow{E}}{\partial t} + \varepsilon_\infty \tau_1\tau_2 \frac{\partial^2 \overrightarrow{E}}{\partial t^2} \right]
\]
(3)

In equation (3), one should note that the total electric field vector has been replaced by its two terms representing the incident and the scattered fields. Similarly, Maxwell’s equations in equation (1) can be re-written as
\[
\frac{\partial \vec{H}}{\partial t} = -\frac{1}{\mu} \nabla \times \vec{E}, \tag{4}
\]
\[
\frac{\partial \vec{D}}{\partial t} = \nabla \times \vec{H} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}. \tag{5}
\]

where \(\nabla \times \vec{H}_i\) is being replaced by \(\varepsilon_0 \frac{\partial \vec{E}_i}{\partial t}\).

Now it is obvious that the vector differential equations (3) to (5) are expressed in terms of incident and scattered fields. The incident field and its derivatives are usually defined analytically. Using the central difference approximation for the derivatives in equations (3-5), one can easily obtain updating equations for the components of the field vectors \(\vec{E}_s\), \(\vec{H}_s\) and the auxiliary displacement vector \(\vec{D}\). The solution to the scattered field components starts by the evaluation of the scattered electric field from (3), then the scattered magnetic field from (4) and finally the displacement field from (5). This cycle is then repeated in accordance with the FDTD algorithm. One disadvantage of this method is the requirement for an extra storage for the \(\vec{D}\) and \(\vec{E}_s\) vectors due to the second derivatives arising in equation (3).

**Numerical Example**

In order to demonstrate the validity of the above formulation, the computation of the reflection from an air water interface due to a wide band Gaussian pulse is performed. The complex permittivity of water is described by a first order Debye equation

\[
\varepsilon^*(\omega) = \varepsilon_0 \left[ \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1 + j\omega\tau_0} \right]
\]

where \(\varepsilon_s = 81\), \(\varepsilon_\infty = 1.8\), and \(\tau_0 = 9.4 \times 10^{-12}\). The one dimensional problem space had 1000 cells with the first 500 cells simulating the free space region and the other 500 cells representing the water half space. The cell size was 37.5 \(\mu m\) and the time step was 0.0625 ps. The incident Gaussian pulse supports frequencies up to 100 GHz. In this one dimensional problem, only \(\vec{E}_{yx}\), \(\vec{H}_{xz}\), \(\vec{H}_{yz}\), and \(\vec{D}_y\) components are considered. The reflection coefficient at the interface is computed from the transform of the incident and reflected \(y\)-components of the electric field. The exact reflection coefficient is also computed from the analytical solution.

Figure 1 shows the time domain response of the incident and reflected \(y\) components of the electric field at the interface, while Fig. 2 shows the corresponding reflection coefficient. The exact values of the reflection coefficient are in good agreement with those obtained with the proposed scattered field FDTD formulation based on the ADE dispersive media method.
Fig. 1 The incident and scattered electric fields versus time at the water-free space interface.

Fig. 2 The reflection coefficient versus frequency.

Conclusion

This paper has successfully demonstrated and validated the scattered field formulation for the differential equation based method for a dispersive media characterized by the Debye equation with two relaxation constants. A one dimensional reflection coefficient calculation is performed and excellent agreement with the numerical data based on exact solution is obtained. The application of the presented formulation to two and three dimensional applications is straightforward.

References


