Determination of Antenna $Q$ from the Reflection-Coefficient Data

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Abstract

Equivalent circuits containing frequency-variable radiation resistances are postulated for small monopole and small loop antennas above a ground plane. The equivalent circuits are valid below the first natural resonance frequency. By adding an external series inductance or a parallel capacitance, the antenna’s $Q$ is evaluated with the use of an invariant form for the reflection coefficient. The antenna’s $Q$ obtained this way accurately predicts the maximum obtainable bandwidth for either a single- or double-tuned matched antenna. At frequencies above the first resonance, the $Q$ is evaluated by approximating the impedance or the reflection coefficient with polynomials within a narrow frequency range.

Keywords: Low $Q$; data fitting; least squares; finite differences; small antennas; bandwidth prediction; invariant definition of $Q$
1. Introduction

Microwave resonators that are used in filters, oscillators, or in the measurement of material properties are characterized by high $Q$ values, often exceeding values of $10^3$ or $10^4$. The reflection coefficients of such resonators are characterized by perfect circles in the complex plane, the so-called $Q$ circles [1]. An accurate determination of the $Q$-factor value requires applying a data-fitting procedure to the $Q$ circle. This was originally manually done on a Smith chart [2], and later done numerically with the use of a computer [3, 4].

For considerably smaller values of the $Q$ factor – say, below the value of 100 – the circles on a Smith chart become distorted into arcs or loops, so that the assumptions used in determining the high $Q$ factors are not valid anymore. It thus becomes necessary to develop methods of determining the low $Q$ value from any shape of the reflection coefficient as a function of frequency. Examples of devices exhibiting low $Q$ factors are antennas, in particular, antennas that are small in comparison with a wavelength [5]. Knowing the $Q$ value of a particular antenna before even attempting to match it can be valuable a priori information in the design process, because the value of $Q$ determines the maximum theoretical bandwidth that can be obtained after the match. Reference [6] pointed out simple equations that predicted the bandwidths for single and double tuning that can be obtained for a known value of the $Q$ factor, and for a prescribed standing-wave ratio. A slight improvement can be achieved by triple-tuning [7].

The methods to be described here are based on the known behavior of the reflection coefficient as a function of frequency. This reflection coefficient can be obtained either by measurement, using a network analyzer, or by computation, using electromagnetic-simulation software. Two different equivalent circuits will be discussed: one for a short dipole (or monopole) antenna, and another for a short loop antenna. Based on the equations for the equivalent circuits, a system of overdetermined linear equations with real coefficients is developed, and solved by the least-squares procedure. The coefficients obtained are then used for computing the $Q$ factor of the antenna.

It is well known that the monopole becomes resonant when its length is approximately one-quarter wavelength, and the loop antenna achieves resonance when its circumference is about one-half of the wavelength. The equivalent circuits accurately mimic the input reflection-coefficient behavior, starting from the very low frequencies and up to this first natural resonance.

At higher frequencies, the simple equivalent circuits are not adequate anymore. However, it is possible to approximate the input impedance with polynomials, within a narrow range around the frequency of interest, in order to determine the antenna’s $Q$ from these polynomials.

A simpler, alternative procedure will also be described, based on a finite-difference method that likewise provides the antenna’s $Q$ factor.

2. Computational Approach

The $Q$ factor is a real number that specifies how sharp is the resonance of a certain resonator. The subject of this paper is how to determine the unloaded $Q$ factor from a known function of the input reflection coefficient, $\Gamma$, (a complex number). In general, $\Gamma$ is a rational function of the complex frequency, $s = \sigma + j\omega$, which can be represented as

$$\Gamma(s) = \frac{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}$$  \hspace{1cm} (1)

When the coefficients of the polynomials in the numerator and denominator are specified, the function $\Gamma(s)$ is uniquely defined, and can be utilized to determine the corresponding $Q$, as will be shown in the paper. The problem of determining the antenna’s $Q$ factor then reduces to specifying an appropriate equivalent circuit, which can be described as a rational function of the form of Equation (1) such that the least-squares data-fitting procedure can be applied to evaluate the coefficients $a_n$ and $b_n$. With the help of these coefficients, the function $\Gamma(s)$ becomes a smooth and well-behaved function, amenable to having derivatives taken. The value of the unloaded $Q$ factor can then be computed from the invariant formula [8]

$$Q_0 = \frac{f_0 \left| \frac{d\Gamma}{df} \right|}{1 - \left| \Gamma \right|^2} \bigg|_{f=f_0}$$  \hspace{1cm} (2)

where $f_0$ is the resonant frequency of the unloaded resonator.

Instead of expressing the unloaded $Q$ in terms of the reflection coefficient, it is possible to express it in terms of the invariant definitions of the input impedance or admittance [8-10]. We prefer to use the reflection coefficient for two reasons. First, the network analyzer typically provides the output data file for the real and imaginary parts of $\Gamma$ (denoted $S_{11}$), evaluated at a number of equidistant frequencies. Second, for the numerical data-fitting procedure to run smoothly, it is prudent to avoid data that can take very large values. Unlike the impedance or admittance, the reflection coefficient is a bounded function of frequency, the absolute value of which always remains smaller than unity (when only passive networks are involved).

The details of the computational procedure for evaluating the $Q$ factors of a small antenna will be explained next for two
practical situations, namely, for a small dipole and for a small loop. These two examples were selected because only a small number of real coefficients is needed for an accurate representation of the complex function \( \Gamma(f) \), a fact that simplifies the presentation of the computational procedure involved, and because these antennas are devices where it is of advantage to possess a low \( Q \) factor. However, for very low frequencies, the antennas' \( Q \) factors again become very large. The procedures to be described here are nevertheless also valid for those large \( Q \)-factor values.

### 3. Small Dipole Antenna

A dipole antenna reaches its lowest natural resonance when its total length is equal to one-half of the free-space wavelength. For a monopole antenna, the same is true when its length is one-quarter of the free-space wavelength. At frequencies below this resonance, the input-impedance equivalent circuit consists mainly of the large capacitive reactance and a small radiation resistance, as shown in Figure 1a. The input characteristic impedance of the network analyzer is typically \( R_e = 50 \, \Omega \). It is thus convenient to normalize the impedance values to \( R_e \).

The radiation resistance, \( R_a \), of a short dipole antenna is a quadratic function of frequency [11, p. 213]. The capacitive reactance, \( X_C \), is dominant, being many times larger than \( R_a \). The inductive reactance, \( X_L \), is small, but it needs to be added for better agreement with realistic data.

Any \( Q \) factor is specified at the resonant frequency, \( f_0 \), where the reflection coefficient crosses the real axis. At frequencies of interest, the small dipole is not yet resonant, because its input impedance is dominated by capacitive reactance, \( X_C \). The reader may be asking, “How come the papers on small antennas display the antenna’s \( Q \) factor as a continuous function of frequency?” Since sooner or later the input impedance of the small antenna will have to be matched to become mainly real, Collin [12] proposed to define the antenna’s \( Q \) by adding an ideal, lossless, reactive element so that the impedance becomes purely real at the frequency of interest. This is illustrated in Figure 1b, where an external inductance, \( X_0 \), is added to the circuit, so that the total input impedance becomes real at the prescribed frequency, \( f_0 \).

For convenience of the data-fitting numerical procedure, we introduce the normalized frequency, \( \phi \):

\[
\phi = \frac{f}{f_0},
\]
(3)

so that the normalized input impedance, \( z \), becomes

\[
z = d_1 \phi^2 + j \left( d_2 \phi - d_3 \phi^{-1} \right),
\]
(4)

where

\[
d_1 = r = \frac{R_a}{R_e} \bigg|_{f = f_0},
\]

\[
d_2 = x_L = \frac{X_0 + X_L}{R_e} \bigg|_{f = f_0},
\]

\[
d_3 = x_C = \frac{X_C}{R_e} \bigg|_{f = f_0}.
\]

The reflection coefficient becomes

\[
\Gamma_{tot} = \frac{z - 1}{z + 1} = \frac{d_1 \phi^2 - 1 + j \left( d_2 \phi - d_3 \phi^{-1} \right)}{d_1 \phi^2 + 1 + j \left( d_2 \phi - d_3 \phi^{-1} \right)} = u + jv.
\]
(6)

**Figure 1a.** The equivalent circuit of the unloaded monopole antenna.

**Figure 1b.** The antenna augmented with an external series inductance.
Dividing Equation (6) into its real and imaginary parts, we obtain two equations:

\[ d_1 \phi^2 (1-u) + d_2 \phi v - d_3 \phi^{-1} v = 1 + u, \]
\[ -d_1 \phi^2 v + d_2 \phi (1-u) - d_3 \phi^{-1} (1-u) = v. \]

For each frequency, there are two equations like the above. We thus have a system of linear equations of the type

\[ \sum_{i=1}^{3} d_i |e_i| = |h|. \]

The partitioned vectors appearing in Equation (9) are defined as follows:

\[ e_1 = \begin{bmatrix} \phi^2 (1-u) \\ -\phi^2 v \end{bmatrix}, \]
\[ e_2 = \begin{bmatrix} \phi v \\ \phi (1-u) \end{bmatrix}, \]
\[ e_3 = \begin{bmatrix} -\phi^{-1} v \\ \phi (1-u) \end{bmatrix}, \]
\[ h = \begin{bmatrix} 1 + u \\ v \end{bmatrix}. \]

To solve the overdetermined system of Equation (10), one defines matrix \( D \) as follows:

\[ D = (|e_1| |e_2| |e_3|), \]

so that the solution in the least-squares sense is

\[ d = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = (D^*D)^{-1} |h|. \]

Once we know the coefficients \( d_1 \) to \( d_3 \), we can perform the derivative required in Equation (2), evaluated at \( \phi = 1 \):

\[ \frac{d \Gamma}{d \phi}|_{\phi=1} = 2 \frac{2d_1 + j(d_2 + d_3)}{d_1 + 1 + j(d_2 - d_3)}. \]

The absolute value of the above is substituted into Equation (2) and then divided by \( 1 - |\Gamma|^2 \) to get the numerical value of \( Q_0 \), valid at the frequency \( f_0 \).

4. Small Loop Antenna

Figure 2a shows the equivalent circuit of a small loop antenna. The impedance behavior is dominated by the loop’s inductance, \( X_L \), and the radiation resistance, \( R_a \). For a small loop antenna, the radiation resistance is proportional to the fourth power of frequency [11, p. 251]. The presence of any stored electric energy must be modeled by a parallel capacitive reactance, \( X_C \), because the input impedance at very low frequencies must, in the limit, approach a short circuit, so \( X_C \) cannot be in series with \( X_L \), as in Figure 1a. To make the small loop antenna resonant at frequency \( f_0 \), an external lossless capacitive reactance, \(-X_0\), should be added to the circuit, as shown in Figure 2b. The radiation resistance and the reactances are normalized in a similar way as in the previous section. The overall reflection coefficient becomes

\[ \Gamma_{tot} = \frac{d_1 \phi^4 - 1 + d_2 \phi^2 + j(d_3 \phi - d_4 \phi^5)}{d_1 \phi^4 + 1 - d_2 \phi^2 + j(d_3 \phi + d_4 \phi^5)}. \]

Figure 2a. The equivalent circuit of the unloaded loop antenna.

Figure 2b. The antenna augmented by an external parallel capacitance.
By dividing Equation (14) into its real and imaginary parts, we formulate again a linear system of the type in Equation (9). The partitioned vectors appearing in the equation are the following:

\[ |\mathbf{e}_1| = \begin{bmatrix} (1-u)\varphi^4 \\ u\varphi^4 \end{bmatrix}, \]

\[ |\mathbf{e}_2| = \begin{bmatrix} (1+u)\varphi^2 \\ u\varphi^2 \end{bmatrix}, \]

\[ |\mathbf{e}_3| = \begin{bmatrix} u\varphi \\ (1-u)\varphi \end{bmatrix}, \]

\[ |\mathbf{e}_4| = \begin{bmatrix} u\varphi^5 \\ (1+u)\varphi^5 \end{bmatrix}, \]

\[ |\mathbf{h}| = \begin{bmatrix} 1+u \\ v \end{bmatrix}. \]

Note that there are now four coefficients, although there are only three real circuit unknowns: \( R_a \), \( X_L \), and \( X_{Ctot} \). These coefficients provide a good description of \( \Gamma_{tot}(f) \), and we see no problem in using them to obtain the value of \( Q_0 \) by the invariant formula of Equation (2). It will be recalled that the derivative of a rational function is

\[ \left( \frac{a}{b} \right)' = \frac{ba' - ab'}{b^2}. \]  

(16)

Although the numerator and the denominator have to be differentiated separately and then these factors combined according to Equation (16), all these steps are easily programmed on a computer.

5. Finite-Difference Procedure

The input data file that contains values of the reflection coefficient as a function of frequency is usually stored with data at equidistant frequency intervals, \( \Delta f' \). Suppose that the \( i \)th point of the data file denotes the desired frequency, \( f_0 \). The value of \( Q_0 \) is then obtained by the second-order finite-difference approximation as follows:

\[ Q_0(i) = f(i) \frac{\Gamma_{tot}(i+1) - 2\Gamma_{tot}(i) + \Gamma_{tot}(i-1)}{\Delta f'} \frac{1}{1 - \Gamma_{tot}(i)} . \]  

(17)

Because of the limited number of digits with which the data are expressed, as well because of any other approximations involved in obtaining the measured or simulated data, the results of Equation (17) are not as accurate as the results obtained by the least-squares procedure. However, when the procedure based on Equation (17) is repeated for several neighboring values of the index \( i \), the resultant value of \( Q_0 \) is not supposed to appreciably change. We found that if the finite-difference procedure is applied to, say, five neighboring points, the mean value of these five results yields \( Q_0 \) within a few percent of the value obtained by the analytic differentiation described in the previous sections. Therefore, it is convenient to evaluate \( Q_0 \) by both methods to check whether they provide similar results. Another advantage of using the finite-difference procedure is that we can interpret the standard deviation of these five points to indicate the standard error of \( Q_0 \), albeit with a pessimistic estimate.

6. Examples

6.1 Spherical-Cap Dipole

Theoretically, dipole antennas can achieve a value of \( Q_0 \) lower than 10 [13]. Lopez [6] described a small dipole, loaded with a spherical cap, which he matched (either single-tuned or double-tuned) to a bandwidth corresponding to \( Q_0 = 19 \). For the frequency range 286.5 MHz to 313.5 MHz, the input reflection coefficient of this antenna was plotted in Lopez’s Figure 2b.

We processed these reflection-coefficient data two times: first, for the equivalent circuit in Figure 1a, and then again for the equivalent circuit shown in Figure 1b. When the coefficients \( d_i \) \( (i = 1 \text{ to } 3) \) were evaluated for the first time, the reflection coefficient, \( \Gamma_a \), was evaluated by Equation (6). Figure 3 shows

![Figure 3. The reflection coefficients of the unloaded and augmented spherical-cap antenna.](image)

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Lopez’s original points, shown as circles, and the recomputed points obtained from Equation (6), shown as + symbols. At first glance, the two data sets agree well with each other. The actual distances between the data are plotted in Figure 4. It can be seen that the overall agreement with the model from our Figure 1a was about one-half percent.

An advantage of our representation in Equation (6) is that it is continuous. It is therefore possible to accurately predict the value of the external reactance, $X_0$, at the frequency of interest, even if that frequency does not coincide with one of the data points. For a target frequency $f_0 = 300$ MHz, the normalized coefficient was found to be $d_2 = x_0 = 2.74$.

According to Figure 1b, this reactance was added to the antenna’s reactance. The normalized impedance was obtained from the reflection-coefficient data as

$$z_a(\phi) = \frac{1 + \Gamma(\phi)}{1 - \Gamma(\phi)}. \quad (18)$$

The impedance was then augmented by the external inductance, as follows:

$$z_{\text{tot}}(\phi) = z_a(\phi) + jx_0\phi, \quad (19)$$

and the result was transformed back into the reflection coefficient,

$$\Gamma_{\text{tot}}(\phi) = \frac{z_{\text{tot}}(\phi) - 1}{z_{\text{tot}}(\phi) + 1}. \quad (20)$$

The data-fitting procedure was repeated for the augmented reflection coefficient using Equation (20), and the new coefficients, $d_i \ (i = 1 \ to \ 3)$, were found. The values of $\Gamma_{\text{tot}}$ were now centered on the real axis. As seen in Figure 4, the data obtained by Equation (20) still agreed with the data obtained from the new coefficients $d_i \ (i = 1 \ to \ 3)$ substituted into Equation (6), but not as perfectly as before. Most actual distances shown in Figure 4 were now smaller than 3% of the Smith-chart radius. Our values of the antenna $Q$ factor were first, as evaluated by data fitting, $Q_0 = 17.9$; and second, as evaluated by finite difference, $Q_0 = 18.0 \pm 1.3$.

It is customary to specify the largest mismatch that can be tolerated, within the prescribed bandwidth, in terms of the return loss, $RL$, in decibels. The return loss is related to the absolute value of the reflection coefficient, $\Gamma$, as follows:

$$\rho = |\Gamma| = 10^{\frac{-RL}{20}}. \quad (21)$$

The theoretical relative bandwidth that can be achieved for a given $Q_0$ and a given $|\Gamma|$ was found by Wheeler (reference [2] in the Lopez paper). For a single-tuned matching, the relative bandwidth is

$$BW_1 = \frac{2\rho}{Q_0(1 - \rho^2)} \quad (22)$$

whereas for the double-tuned matching, the theoretically possible relative bandwidth is

$$BW_2 = \frac{2\sqrt{\rho}}{Q_0(1 - \rho)}. \quad (23)$$

By substituting $Q_0$ into Equations (22) and (23), it was possible to predict that the spherical-cap antenna could be single-tuned to a 10-dB relative bandwidth of 3.7%, or double-tuned to a relative bandwidth of 9.2%, even before the actual match was attempted. These values agreed well with Lopez’s results.

### 6.2 Loop Antenna Above the Ground Plane

Figure 5 shows a printed-circuit loop antenna on a ground plane. The loop consisted of a 12 mm-wide strip with outside dimensions of $104 \times 104$ mm. The substrate was 1.5 mm thick, and its relative dielectric constant was $\varepsilon_r = 2.5$. As this antenna could be contained within a hemisphere of radius $a = 104\sqrt{2} = 147$ mm, it was considered small for $ka < 1$, i.e., up to a frequency of 325 MHz.

The input impedance of the loop antenna in a printed form over a dielectric substrate was simulated using the FDTD method, based on [14]. For computational simplicity, the ground plane was removed, and the input impedance was computed for a loop twice as big in free space. The computed impedance was divided by two to represent the impedance of a loop on an infinite ground plane. The computed values are shown in Figure 6 for the frequency range 100 MHz to 500 MHz.
Theoretically, at zero frequency, the input impedance was a short circuit, which is the left-most point on the real axis of the Smith chart in Figure 6. As frequency grew, the points moved on the periphery of the Smith chart in the clockwise direction. The impedance at first was purely inductive. The lowest computed point shown in Figure 6 was at 100 MHz. As frequency further grew, the points moved in a clockwise manner, and at 248 MHz, they crossed the real axis and became capacitive. This was the first natural resonant frequency of the antenna. Although the impedance was real, the value was very high (about 9 kΩ), and difficult to match to 50 Ω. After that, the points became capacitive in nature, and moved inward on the Smith chart.

The equivalent circuit from Figure 2a was valid for the upper half of the Smith chart. Figure 7 shows the difference between the reflection coefficient computed by the circuit in Figure 2a, and the input reflection coefficient obtained by FDTD. Over the entire range of 100 MHz to 250 MHz, the difference was less than 1% of the radius of the Smith chart. At any frequency of interest within this range, one could find the required value of the parallel susceptance that would bring the antenna to resonance at this frequency. For instance, at \( f_0 = 200 \) MHz, an external parallel capacitance of \( C_0 = 1.5 \) pF would cancel the antenna’s susceptance, so that the \( Q \) factor obtained from Equation (2) was found to be \( Q_0 = 103.2 \). This was already a disappointingly high antenna \( Q \), but the value of the input resistance obtained by this external “matching” capacitance was even more disappointing: \( R_{in} = 20 \) kΩ.

Above the first natural resonance frequency, the equivalent circuit in Figure 2a is no longer appropriate for accurately simulating the input impedance of the unloaded loop antenna. We used polynomial approximations of order three to separately represent the real and the imaginary parts of the reflection coefficient, and to find the analytical value of \( Q_0 \) with the use of Equation (2). This works only in a relatively narrow range in the vicinity of frequency \( f_0 \). As long as the results agree with the finite-difference results from Equation (17), we feel confident in the values obtained.

Table 1 summarizes the results of determining \( Q_0 \) for the small loop antenna. \( BW_1 \) and \( BW_2 \) are the single-tuned and double-tuned 10 dB bandwidths computed by using Equations (22) and (23), using \( Q_0 \) as listed in the second column of the table. \( R_{in} \) is the input resistance of the antenna, augmented by a series inductance that made the antenna resonate at \( f_0 \). At frequencies below the first natural resonance (248 MHz), the bandwidths were too narrow, and the “matched” input resistance, \( R_{in} \), was too high for practical applications.

At \( f_0 = 325 \) MHz, the loop antenna is still considered “small,” and at this frequency, the antenna’s \( Q \) was relatively low, namely \( Q_0 = 17.5 \), while \( R_{in} \) was close to 50 Ω. At \( f_0 = 335 \) MHz, \( R_{in} \) became practically equal to 50 Ω. The required external series inductance at this frequency was \( L_0 = 148 \) nH. As can be seen from Figure 8, the input imped-
Table 1. The loop antenna’s $Q$ and associated bandwidths.

<table>
<thead>
<tr>
<th>$f_0$ (MHz)</th>
<th>$Q_0$ using (2)</th>
<th>$Q_0$ using (17)</th>
<th>$BW_1$ (MHz)</th>
<th>$BW_2$ (MHz)</th>
<th>$R_{in}$ (Ω)</th>
</tr>
</thead>
<tbody>
<tr>
<td>105</td>
<td>1228</td>
<td>1234±17</td>
<td>0.06</td>
<td>0.14</td>
<td>130k</td>
</tr>
<tr>
<td>200</td>
<td>103.2</td>
<td>107±9</td>
<td>1.3</td>
<td>3.2</td>
<td>20k</td>
</tr>
<tr>
<td>248</td>
<td>39.5</td>
<td>39±1</td>
<td>4.2</td>
<td>10.3</td>
<td>8.7k</td>
</tr>
<tr>
<td>325</td>
<td>17.5</td>
<td>18±0.1</td>
<td>12.4</td>
<td>30.5</td>
<td>54.5</td>
</tr>
<tr>
<td>335</td>
<td>15.9</td>
<td>16±0.3</td>
<td>14.1</td>
<td>34.7</td>
<td>50.1</td>
</tr>
<tr>
<td>400</td>
<td>8.38</td>
<td>8.5±0.1</td>
<td>31.8</td>
<td>78.5</td>
<td>43.9</td>
</tr>
</tbody>
</table>

Figure 8. At $f_0 = 335$ MHz, the series inductance matched the antenna to 50 Ω. The ○ symbols denote the computed input impedance, and the + symbols denote the polynomial approximation.

Figure 9. The magnitude of the reflection coefficient, based on the data computed by the FDTD.

Table 1 also shows that the antenna $Q$ of the loop antenna from Figure 5 could be as low as $Q_0 = 8.38$ at the somewhat higher frequency of $f_0 = 400$ MHz.

7. Conclusions

Simple equivalent circuits have been verified to fit the reflection-coefficient data obtained by electromagnetic-simulation software well. An example of the spherical-cap dipole was simulated in the literature by the Moment Method, and the printed-circuit loop antenna was simulated here by the FDTD method. In both cases, it was found that the input impedances, based on the equivalent circuits, agreed well with those values from the software simulations, starting from very low frequencies up to the first natural resonance of the unloaded antenna.

At frequencies higher than the first resonance, the simple equivalent circuits were not adequate. We used polynomial approximations for the real and imaginary parts of the reflection coefficient. Once the reflection coefficient was expressed with continuous functions, the invariant forms of the unloaded $Q$ factor provided good estimates of the antenna’s $Q$. Although the invariant forms are also known for the impedance and the admittance, we preferred to use the invariant form for the reflection coefficient.

8. References


